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Algorithm of the Parallel Sweep Method for Numerical Solution of the Gross–Pitaevskii Equation with Highest Nonlinearities

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In this paper, we for the first time introduce a numerical scheme the solution of a nonlinear equation of the Gross–Pitaevskii type (GP) or the nonlinear Schrodinger equation (NLSE) with highest nonlinearities, which provides implementation of a complete set of motion integrals. This scheme was parallelly implemented on a non-uniform grid. Propagation of a ring laser beam with non-zero angular momentum in the filamentation mode is studied using the implemented numerical scheme. It is shown, that filaments under exposure to centrifugal forces escape to the periphery. Based on a number of numerical experiments, we have found the universal property of motion integrals in the non-conservative case for a given class of equations. Research of dynamics of angular momentum for a dissipative case are also presented. We found, that angular moment, particularly normed by initial energy during filamentation process, is quasi-constant.

Keywords: nonlinear Schrodinger equation, fast parallel algorithm, fully conservative numerical scheme, motion integral.

Introduction

A nonlinear parabolic partial differential equation (or a system of such equations) occurs in many applications [3]. In such equations, their rigorous analytical solutions are often unknown. Generally, such equations are solved by numerical methods. Correctness of application of these numerical methods has to be controled, possibly, by comparing the solutions obtained with the known rigorous properties of these equations. Apparently, for the whole branch of these studies, such stage has been completed [2]. Indeed, inspite of the fact that for the GP (NLSE) equation with highest nonlinearities such properties are known, application checks of these properties were not carried out anywhere, except for [2]. One of special cases of this class of equations is the nonlinear Schrodinger equation with highest nonlinearities. In [2], a wide range of numerical schemes used for solution of the GP (NLSE) equation with highest nonlinearities was constructed by the example of a case with radial symmetry, and it was determined that the simplest and quite effective numerical method is the method of splitting by physical factors method. At the same time, discrete difference methods for solving the NLS equation are optimal for tracking and suppressing numerical imbalances, and the adaptive step along the evolutionary coordinate should be selected according to the conditions of preserving the Hamilton function on the numerical solution of the GP (NLSE) equation. Usually this step is significantly less compared to those offered in other works. In case of implementing this requirement on a grid in 2D+1 dimensions, development of methods for numerical solution on the non-uniform grids with the used parallelization methods becomes urgent. In this paper, we solved this problem.

1. The GP (NLSE) Equation and its Exact Properties

In this paper, we numerically explore the behavior of solutions of the complex Gross – Pitaevskii (GP) (or non-linear Schrodinger equation (NLSE)) equation with higher nonlineari-

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ties [1]:

$$\partial_t\psi + i(\epsilon(\psi) - \partial_\mu\partial^\mu - i\alpha(\psi)/2)\psi = 0, \tag{1}$$

where x^μ are transverse coordinates (in this article, the situation is considered as $\vec{x} \in \mathbb{R}^2$); $\psi(x)$ is complex-valued function, which, depending on the context, can have a different physical meaning; $\epsilon(\phi)$ is nonlinear function in the simplest case of a cubic in the field, but we will consider a more complicated situation with higher nonlinearities; $\alpha(\psi)$ is the function of nonlinear absorption. We will discuss three conservation laws that correspond to three global symmetries: the outer conservation is a shift in the evolutionary coordinate $t \to t + a$ (H-energy), and the inner law is the phase shift of the complex field $\psi \to \psi e^{ia}$ (the E-number of a particle), and the symmetry with respect to turn transformation ($y_\mu = A^\nu_\mu x_\nu$ Where $A \in so2$) (M-angular momentum). Due to the fact that we consider a model with dissipation of the ratio to be generalizing, the known conservation laws will take the following form:

$$\frac{\partial H}{\partial t} = -\int (i\alpha(\psi\Phi^*_\psi - \psi^*\Phi_\psi))d\vec{x}, \tag{2}$$

where $H \equiv \int (\partial_\mu\psi^*\partial^\mu)\psi + F_\epsilon)d\vec{x}$ is the Hamilton function; $\epsilon \equiv \delta F_\epsilon(\phi)/\delta\phi$ is the power in terms of mechanics or the nonlinear additive to the refractive index of the terms of optics; $\Phi_\psi = \delta H/(\delta\psi)$

$$\frac{\partial}{\partial t}\int \psi\psi^* d\vec{x} = -\int \alpha\psi\psi^* d\vec{x}, \tag{3}$$

$$\frac{\partial}{\partial t}\int m d\vec{x} = -\int \alpha m d\vec{x}, \tag{4}$$

where $m \equiv \vec{P} \times \vec{x}$ is the density of angular momentum; $P_\mu = (\phi\partial_\mu\psi^* - \psi^*\partial_\mu\psi)/2i$ is the Poynting vector. Implementation of the given exact relations in the numerical solution, even in the conservative case, imposes very strict conditions on the numerical grid, which makes the use of parallel algorithms to be relevant.

2. The Numerical Scheme

Taking into account the initial conditions, we construct an inhomogeneous grid which is origin-symmetric with the distance between the nodes increasing according to the law of geometric progression x_I. The same way, we will implement splitting of the orthogonal coordinate y_I. For simplicity, assume that each processor has the same number of points. The following relation is binding global index J to local index j, which is localized on the processor with number q:

$$J = (q-1) * m_l + j. \tag{5}$$

We will enable numerical implementation of a step of diffraction on a three-point "cross" scheme with the second order of accuracy of approximation by the Laplace operator on the non-uniform grid. In this case, we need to solve a system of equations of the form:

$$a_J\psi^{l+1}_{J-1} - c_J\psi^{l+1}_J b_J\psi^{l+1}_{J+1} = -f^l_J. \tag{6}$$

We generalize the technique of fast parallelizing [4] to a complex case.

STATEMENT: Solution of a complex-valued equation can be found by the following relation:

$$\psi_J = g_{q-1}u_j + g_q\nu_j + w_j. \tag{7}$$

Here $a_j u_{j-1} - c_j u_j + b_j u_{j+1} = 0$, where conditions are met at the borders $u_{(q-1)m_l} = (1,0); u_{qm_l} = (0,0)$, $a_j\nu_{j-1} - c_j\nu_j + b_j\nu_{j+1} = 0$ and $\nu_{(q-1)m_l} = (0,0); \nu_{qm_l} = (1,0)$. And finally, we give the equations for the internal field $a_j w_{j-1} - c_j w_j + b_j w_{j+1} = -f_j$. Where conditions are met at the borders: $w_{(q-1)m_l} = (0,0); w_{q(m_l)} = (0,0)$. Here, crosslinking coefficients g_q can be found by the following equation:

$$A_q g_{q-1} - C_q g_q + B_q g_{q+1} = -F_q, \tag{8}$$

$$-C_0 g_0 + B_0 g_1 = -F_0, \tag{9}$$

$$A_M g_{M-1} - C_M g_M = -F_M, \tag{10}$$

where $C_0 = c_0 - b_0 u_1$, $B_0 = b_0 * \nu_1$,$F_0 = f_0 - b_0 w_1$, $A_q = a_{qm_l} u_{q(m_l-1)}$, $C_q = -a_{qm_l}\nu_{q*(m_l-1)} + c_{qm_l} - b_{qm_l} u_{j(m_l+1)}$, $B_q = b_{qm_l} u_{q(m_l-1)}$, $F_q = f_{qm_l} + a_{qm_l} w_{q(m_l-1)} - b_{qm_l} w_{q(m_l+1)}$, $q = 1..M-1$, $A_M = a_{Mm_l} u_{M(m_l-1)}$, $C_M = -a_{Mm_l}\nu_{M(m_l-1)} - b_{Mm_l} u_{j(m_l+1)}$, $F_M = f_{Mm_l} + a_{Mm_l} w_{M(m_l-1)}$.

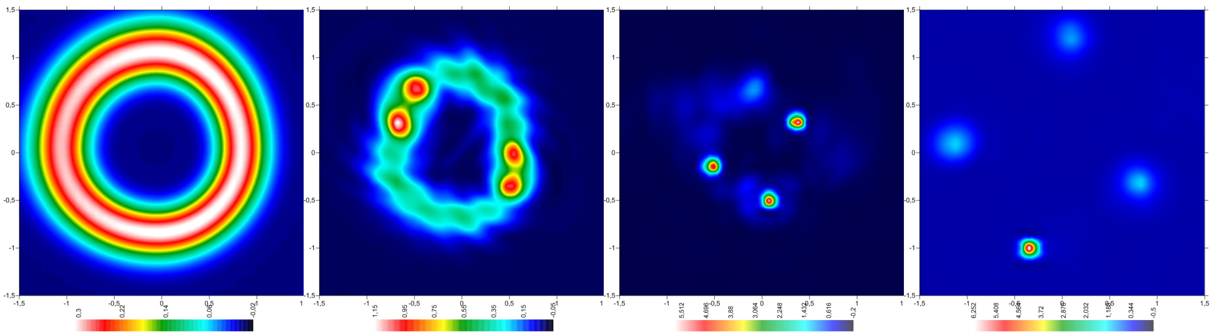
The proof of these results can be provided by direct substitution.

2.1. Results of Numerical Calculations

Results of numerical calculation for a beam with initial condition of the form are given below (in a radial coordinate system):

$$\psi_0(r,\varphi) = (\exp(-0.5(r^2)) - \exp(-2.5(r^2))) * \exp(-i\varphi m_t) * f_{noise}, \tag{11}$$

where m_t is the topological charge (in our case, $m_t = 2$); $n_2 = 35$, $K = 8$. The remaining nonlinearity parameters were chosen in the same way as in [5]. Where f_{noise} is the noise component, which violates the central symmetry.



z=0 z=0.7 z=1 z=1.6

Figure 1. Distribution of the intensity field at different points of the propagation distance

Examples of distributions $|\psi|^2$ for different distances are shown in Fig.1. This example demonstrates that compression of beam in a result of the absorption effects has stopped, and, eventually, the beam's divergence is completely realized as it takes place and the similar radiative case [5]. A series of numerical experiments were made. The obtained results allow extending the findings in [5] on the general case. Namely, as we can see from numerical calculations, value ΔE is proportional to $\Delta\bar{H}$ with high degree of accurasy, i.e. the following relation is true:

$$H(z) = H(0) + \gamma\Delta E, \tag{12}$$

in addition, it was determined that

$$\bar{M} \equiv \int m d\vec{x}/E \approx const. \tag{13}$$

Conclusion

A parallel scheme for numerical solution of the NSE on an irregular grid using the accurate method for the system of solving of linear equations is proposed. This case is a tridiagonal system of linear algebraic equations which arises due to a discrete difference approximation of the GP (NLSE). Generalization of the Yanenko method to the complex case is proposed. By the example of numerical solution of the GP (NLSE) with topological charge, it is found that in case of absorption, the linear combination of the number of particles and energy is a constant value; the normalized number of particles of angular momentum is conserved with high accuracy.

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